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Finiteness of R-equivalence groups of some adjoint classical groups of type ${}^2D_3$

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Abstract

Let F be a field of characteristic different from 2. We construct families of adjoint groups G of type ${}^2D_3$ defined over F (but not over k) such that $G(F)/R$ is finite for various fields F which are finitely generated over their prime subfield. We also construct families of examples of such groups G for which $G(F)/R \simeq \mathbb{Z}/2\mathbb{Z}$ when $F = k(t)$, and k is (almost) arbitrary. This gives the first examples of adjoint groups G which are not quasi-split nor defined over a global field, such that $G(F)/R$ is a non-trivial finite group.

Keywords: R-equivalence group; Classical adjoint groups; Unramified cohomology

Introduction

For an algebraic group G defined over a field F, let $G(F)/R$ be the group of R-equivalence classes introduced by Manin in [10]. The algebraic group G is called R-trivial if $G(L)/R = 1$ for every field extension L/F. It was established by Colliot-Thélène and Sansuc in [4] (see also [11, Proposition 1]) that the group G is R-trivial if the variety of G is stably rational. Moreover, in [4], the following question was raised:

Question. Let F be a field which is finitely generated over its prime subfield, and let G be a connected linear algebraic group defined over F. Assume that F is perfect or G is reductive. Is $G(F)/R$ finite?

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The question was answered positively by Colliot-Thélène and Sansuc if G is quasi-split (cf. [4, Proposition 14]) and by Gille for any reductive group G defined over a global field in [5]. Lemma II.1.1(c) of [5] immediately implies that this question has a positive answer if F is a rational extension of a global field k and G is defined over k. Various examples of classical adjoint groups which are not R-trivial were constructed in [1] or [6], [11]. Throughout this paper, we will assume that F is a field of characteristic different from 2 and we will focus on absolutely simple adjoint groups of type ${}^2D_3$. If F/k is a finitely generated field extension, we construct an infinite family of adjoint groups G of type ${}^2D_3$ defined over F such that $G(F)/R$ is finite as soon as $H^3_{nr}(F/k,\mu_2)$ is finite. If $F=k(t)$, where k is an arbitrary field, we will also give a family of examples of such groups for which $G(F)/R\simeq\mathbb{Z}/2\mathbb{Z}$. This gives the first examples of adjoint groups G such that $G(F)/R$ which are not quasi-split nor defined over a global field, such that $G(F)/R$ is a non-trivial finite group.

1. Unramified cohomology

Let X be a smooth proper irreducible variety defined over k. We denote by $X^{(1)}$ the set of points of codimension 1 in X. The ring $\mathcal{O}_{X,x}$ is then a discrete valuation ring. We will denote by υ_x the corresponding discrete valuation and by π_x a local parameter. We have a residue map

$$\partial_x : H^n\big(k(X),\mu_2\big)\to H^{n-1}\big(\kappa(x),\mu_2\big),$$

where $\kappa(x)$ denotes the residue field $\mathcal{O}_{X,x}/(\pi_x)$. If $u\in\mathcal{O}_{X,x}$, we will denote by $\bar{u}$ its image in $\kappa(x)$.

The residue of a cohomology class $\alpha\in H^n(k(X),\mu_2)$ can be computed as follows: denote by $k(X)_x$ the completion of $k(X)$ with respect to the valuation on $\mathcal{O}_{X,x}$. Then π_x is also a local parameter for the unique discrete valuation on $k(X)_x$ extending υ_x, and we have an injection $H^n(\kappa(x),\mu_2)\hookrightarrow H^n(k(X)_x,\mu_2)$. Then we have a decomposition

$$\mathrm{Res}_{k(X)_x/k(X)}(\alpha)=\alpha_0+(\pi_x)\cup\alpha_1,$$

for some uniquely determined $\alpha_i\in H^{n-i}(\kappa(x),\mu_2)$. We then have the equality $\partial_x(\alpha)=\alpha_1$. In particular, for every $a_1,\dots,a_{n-1},b_1,\dots,b_{n-1}\in\mathcal{O}^\times_{X,x}$, we have

$$\begin{aligned}&\partial_x\big((\pi_x)\cup(a_1)\cup\cdots\cup(a_{n-1})\big)=(\bar{a}_1)\cup\cdots\cup(\bar{a}_{n-1}),\\ &\partial_x\big((b_1)\cup\cdots\cup(b_n)\big)=0.\end{aligned}$$

We say that $\alpha\in H^n(k(X),\mu_2)$ is *unramified at* x if $\partial_x(\alpha)=0$. In this case, the class α_0 is called *the specialisation of* α *at* x, and is denoted by $s_x(\alpha)$. It does not depend on the choice of π_x. If $\partial_x(\alpha)\neq 0$, we say that α is *ramified at* x, and that x is *a pole* of α. It is well known that the set of poles of α is finite. The *unramified cohomology group* $H^n_{nr}(k(X),\mu_2)$ is the subgroup of $H^n(k(X),\mu_2)$ consisting of classes which are unramified at every $x\in X^{(1)}$. It is a birational invariant of X. In particular, if X is a rational variety, then the restriction map induces an isomorphism $H^n(k,\mu_2)\simeq H^n_{nr}(k(X),\mu_2)$. Therefore if F/k is a finitely generated extension, we can define the group of unramified elements $H^n_{nr}(F/k,\mu_2)$ by

$$H^n_{nr}(F/k,\mu_2)=H^n_{nr}\big(k(X),\mu_2\big),$$

where X is any irreducible smooth proper model of F/k. We refer to [2] for more details.

Notice that for any finitely generated field extension F/k, the elements lying in the image of $\mathrm{Res}_{F/k} : H^n(k, \mu_2) \to H^n(F, \mu_2)$ are unramified. Such elements are called *constant*. Notice also that if $\alpha \in H^n(F, \mu_2)$ is constant, then we have $s_x(\alpha) = \mathrm{Res}_{\kappa(x)/k}(\alpha)$ for all $x \in X^{(1)}$.

2. *R*-equivalence groups of adjoint groups of type $^2D_3$

2.1. A result of Merkurjev

In this section, we recall Merkurjev's computation of the group of R-equivalence classes of some absolutely simple adjoint classical groups of type $^2D_3$ (cf. [11]). Let (A, σ) be a F-central simple algebra of degree 6 with an orthogonal involution, so we can write $A = M_3(Q)$, where Q is a quaternion F-algebra, and let $\mathbf{PGO}^+(A, \sigma)$ be the connected component of $\mathbf{PGO}(A, \sigma)$, the group-scheme of projective similitudes of (A, σ).

Assume that A is not split, $\mathrm{disc}(\sigma) \in F^\times/F^{\times 2}$ is not trivial, and that the Clifford algebra $C(A, \sigma)$ has index 2. If $L = F(\sqrt{\mathrm{disc}(\sigma)})$ then A_L (or equivalently Q_L) is split. Hence we can write $Q \simeq (\mathrm{disc}(\sigma), \alpha)$, for some $\alpha \in F^\times$. Let $1, i, j, ij$ be the corresponding standard basis for Q, and let γ be the canonical (symplectic) involution on Q. The involution σ is adjoint to a skew-hermitian form (V, h) over (Q, γ), where V is a right Q-vector space of dimension 3.

The skew-hermitian form h represents xi for some $x \in F^\times$, so we can write $h = h' \perp \langle xi \rangle$ for some skew-hermitian form (V', h') over (Q, γ) of trivial discriminant, where V is a right Q-vector space of dimension 2.

Set $(A', \sigma') := (\mathrm{End}_Q(V'), \sigma_{h'})$. Then $C(A', \sigma') = Q_1 \times Q_2$, for some quaternion F-algebras Q_1 and Q_2 satisfying $Q_1 \otimes Q_2 = Q$ in $\mathrm{Br}(F)$. Moreover, $(Q_1)_L \simeq (Q_2)_L$ and $C(A, \sigma) = (Q_i)_L$ in $\mathrm{Br}(L)$ (so $(Q_i)_L$ is not split for $i = 1, 2$).

Proposition 1. *Under the previous notation, we have the following group isomorphism*:

$$\mathbf{PGO}^+(A, \sigma)(F)/R \simeq N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_1}(Q_1^\times) \cdot \mathrm{Nrd}_{Q_2}(Q_2^\times)/N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_i}(Q_i^\times).$$

For a proof of all these facts, see [11, Section 3]. Notice that in [11], Merkurjev described more generally the group $G(F)/R$, when G is an absolutely simple adjoint classical group defined over F.

2.2. Finiteness of some *R*-equivalence groups

2.2.1. Some useful lemmas

We will assume that (A, σ) is as in the previous section. We start to investigate the finiteness of $\mathbf{PGO}^+(A, \sigma)(F)/R$. Keeping the notation above, we will identify this group to

$$N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_1}(Q_1^\times) \cdot \mathrm{Nrd}_{Q_2}(Q_2^\times)/N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_i}(Q_i^\times).$$

If $\lambda \in N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_1}(Q_1^\times) \cdot \mathrm{Nrd}_{Q_2}(Q_2^\times)$, we will denote by $[\lambda]$ its class modulo $N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_i}(Q_i^\times)$. We start with an easy lemma:

Lemma 2. *Let F be any field of characteristic different from 2. Then the map*

$$\varphi : \mathbf{PGO}^+(A, \sigma)(F)/R \to H^3(F, \mu_2), \quad [\lambda] \mapsto (\lambda) \cup [Q_1]$$

is a well-defined injective group homomorphism.

Proof. Since $(\mathrm{Nrd}_{Q_1}(Q_1^\times)) \cup [Q_1] = 0$, this map is a well-defined group homomorphism. If $\lambda \in N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_1}(Q_1^\times) \cdot \mathrm{Nrd}_{Q_2}(Q_2^\times)$ satisfies $(\lambda) \cup [Q_1] = 0$, then $\lambda \in \mathrm{Nrd}_{Q_1}(Q_1^\times)$ by a well-known theorem of Merkurjev [12], so $[\lambda] = 1$. □

Remark 3. In view of this lemma, we just have to investigate the finiteness of the image of φ.

We now assume until the end that X is a smooth irreducible proper model of F defined over k.

Lemma 4. *Assume that Q_1 and Q_2 have no common pole, and let $x \in X^{(1)}$. Then*

$$\partial_x\big((\lambda) \cup [Q_1]\big) = \begin{cases} 0 & \text{if } x \text{ is not a pole of } [Q_1] \text{ or } [Q_2], \\ 0 \text{ or } s_x[Q_2] & \text{if } x \text{ is a pole of } [Q_1], \\ 0 \text{ or } s_x[Q_1] & \text{if } x \text{ is a pole of } [Q_2]. \end{cases}$$

Proof. Notice that since $\lambda = N_{L/F}(z)$ for some $z \in L^\times$ and that $(Q_1)_L \simeq (Q_2)_L$, we get

$$(\lambda) \cup [Q_1] = \mathrm{Cor}_{L/F}\big((z) \cup [Q_1]_L\big) = \mathrm{Cor}_{L/F}\big((z) \cup [Q_2]_L\big) = (\lambda) \cup [Q_2].$$

Let $x \in X^{(1)}$, and assume first that $[Q_1]$ and $[Q_2]$ are both unramified at x. If (λ) is unramified at x, then $(\lambda) \cup [Q_1]$ is also unramified at x, that is $\partial_x((\lambda) \cup [Q_1]) = 0$. If (λ) is ramified at x, then write $\lambda = \lambda_1\lambda_2, \lambda_i \in \mathrm{Nrd}_{Q_i}(Q_i^\times)$. Then (λ_1) or (λ_2) is ramified at x, since $\partial_x((\lambda)) = \partial_x((\lambda_1)) + \partial_x((\lambda_2))$ and $\partial_x((\lambda_i)) \in \mathbb{Z}/2\mathbb{Z}$. If (λ_2) is unramified at x, then $\partial_x((\lambda) \cup [Q_1]) = \partial_x((\lambda_2) \cup [Q_1]) = 0$. Now assume that (λ_2) is ramified at x, so (λ_1) is unramified at x. Since $[Q_2]$ is unramified at x as well, then $\partial_x((\lambda) \cup [Q_1]) = \partial_x((\lambda) \cup [Q_2]) = \partial_x((\lambda_1) \cup [Q_2]) = 0$. Hence $\partial_x((\lambda) \cup [Q_1]) = 0$ if x is not a pole of $[Q_1]$ or $[Q_2]$.

Now assume that x is a pole of $[Q_1]$, so $[Q_2]$ is unramified at x by assumption. If (λ) is unramified at x then $\partial_x((\lambda) \cup [Q_1]) = \partial_x((\lambda) \cup [Q_2]) = 0$. If (λ) is ramified at x, then $\partial_x((\lambda) \cup [Q_1]) = \partial_x((\lambda) \cup [Q_2]) = s_x([Q_2])$. If x is a pole of $[Q_2]$, then similar computations show that $\partial_x((\lambda) \cup [Q_1]) = 0$ or $s_x([Q_1])$. □

2.2.2. The case where $H^3_{nr}(F/k, \mu_2)$ is finite

Proposition 5. *Assume that $[Q_1]$ and $[Q_2]$ have no common pole. If $H^3_{nr}(F/k, \mu_2)$ is finite, then* $\mathbf{PGO}^+(A, \sigma)(F)/R$ *is finite.*

Proof. By assumption, the kernel of the map

$$(\partial_x)_{x \in X^{(1)}} : \mathrm{Im}(\varphi) \to \prod_{x \in X^{(1)}} H^2\big(\kappa(x), \mu_2\big)$$

is finite. By the previous lemma, its image is finite as well, so we are done by Remark 3. □

Examples 6. The group $H^3_{nr}(F/k, \mu_2)$ is finite in the following cases (and therefore the previous proposition may be applied):

(1) $H^3(k, \mu_2)$ is finite and X is a smooth conic over k;
(2) k is a finite field and X is a smooth proper variety of dimension 2 over k;

(3) k is either a local field (i.e. a finite extension of $\mathbb{Q}_p$), $\mathbb{R}$ or $\mathbb{C}$ and X is a smooth proper geometrically irreducible curve over k;
(4) k is a number field and X is a smooth proper geometrically irreducible curve over k.

Case (1) readily follows from Propositions 3 and A.1 of [7]. Case (2) follows from Theorem 0.8 of [8]. Now let us consider case (3): if k is a local field, it follows from Corollary 2.9. of [8]. If $k=\mathbb{R}$, it follows from a result of Colliot-Thélène and Parimala (see [3]). Finally, if $k=\mathbb{C}$, then $k(X)$ has cohomological dimension at most 1 and therefore $H^3(k(X),\mu_2)=0$. In case (4), it readily follows from Theorem 0.8 of [8] that we have an injective homomorphism

$$H^3_{nr}\big(k(X),\mu_2\big)\hookrightarrow\prod_{\upsilon\in P(k)}H^3_{nr}\big(k_\upsilon(X),\mu_2\big),$$

where $P(k)$ denotes the set of all places of k. By Corollary 2.9 of [8], $H^3_{nr}(k_\upsilon(X),\mu_2)$ is zero if X has good reduction with respect to υ. Since X has good reduction with respect to all but finitely many places, it follows from case (3) that $H^3_{nr}(k(X),\mu_2)=H^3_{nr}(F/k,\mu_2)$ is finite.

The reader may find more finiteness results for $H^3_{nr}(F/k,\mu_2)$ in [2].

2.2.3. The case where $H^3_{nr}(F/k,\mu_2)\simeq H^3(k,\mu_2)$

We give here another family of examples. Keeping notation of the previous sections, we will assume that Q_1 and Q_2 have no common poles. We then set

$$S_1=\big\{x\in X\mid x\text{ is a pole of }Q_2\text{ such that }s_x\big([Q_1]\big)\neq 0\big\},$$
$$S_2=\big\{x\in X\mid x\text{ is a pole of }Q_1\text{ such that }s_x\big([Q_2]\big)\neq 0\big\}.$$

Proposition 7. *Assume that $[Q_1]$ and $[Q_2]$ have no common pole, and let n_i be the number of elements of S_i. Assume that the restriction map induces a group isomorphism $H^3_{nr}(F/k,\mu_2)\simeq H^3(k,\mu_2)$ (e.g., F/k is rational) and that there exists $x_0\in X^{(1)}$ satisfying the following conditions*:

(1) *One of the class $[Q_i]$ is unramified at x_0 and the corresponding specialisation is zero.*
(2) *The restriction map $\mathrm{Res}_{\kappa(x_0)/k}:H^3(k,\mu_2)\to H^3(\kappa(x_0),\mu_2)$ is injective.*

Then $\mathbf{PGO}^+(A,\sigma)(F)/R$ is finite, and its cardinality is at most $2^{n_1+n_2}$.

Proof. Without any loss of generality, we may assume, for example, that $[Q_1]$ is unramified at $x_0\in X^{(1)}$ and that $s_{x_0}([Q_1])=0$. Assume that $(\lambda)\cup[Q_1]\in\mathrm{Im}(\varphi)$ lies in the kernel of the map

$$(\partial_x)_{x\in X^{(1)}}:\mathrm{Im}(\varphi)\to\prod_{x\in X^{(1)}}H^2\big(\kappa(x),\mu_2\big).$$

By assumption $(\lambda)\cup[Q_1]$ is constant, so we have

$$(\lambda)\cup[Q_1]=\mathrm{Res}_{k(X)/k}(\xi)\quad\text{for some }\xi\in H^3(k,\mu_2).$$

Therefore we get

$$s_x\big((\lambda)\cup[Q_1]\big)=\mathrm{Res}_{\kappa(x)/k}(\xi)\quad\text{for all }x\in X^{(1)}.$$

Since $\partial_{x_0}([Q_1]) = s_{x_0}([Q_1]) = 0$, we have $\mathrm{Res}_{k(X)_{x_0}/k(X)}((\lambda) \cup [Q_1]) = 0$, and therefore $s_{x_0}((\lambda) \cup [Q_1]) = \mathrm{Res}_{\kappa(x_0)/k}(\xi) = 0$. Since the restriction map $\mathrm{Res}_{\kappa(x_0)/k} : H^3(k, \mu_2) \to H^3(\kappa(x_0), \mu_2)$ is injective, we get $\xi = 0$, and thus $(\lambda) \cup [Q_1] = 0$. Therefore $[\lambda] = 1 \in \mathbf{PGO}^+(A, \sigma)(F)/R$ by Lemma 2. It follows that we have an injection

$$\mathbf{PGO}^+(A, \sigma)(F)/R \hookrightarrow \prod_{x \in X^{(1)}} H^2\big(\kappa(x), \mu_2\big).$$

The use of Lemma 4 leads to the conclusion. □

Let us now consider the case where $F = k(t)$, where t is an indeterminate over k, so one may take $X = \mathbb{A}^1_k$. A point x of $\mathbb{A}^1_k$ of codimension 1 then corresponds to a unique monic irreducible polynomial $\pi \in k[t]$ and $\kappa(x) \simeq k[t]/(\pi)$. In this case, we will say that a cohomology class is (un)ramified at π, and ∂_x and s_x will be respectively denoted by ∂_π and s_π. If π has odd degree, a classical restriction–corestriction argument show that the restriction map $H^3(k, \mu_2) \to H^3(k[t]/(\pi), \mu_2)$ is injective. Hence, from the previous proposition, we obtain:

Corollary 8. *Let $F = k(t)$ and assume that $[Q_1]$ and $[Q_2]$ have no common pole. Let n_i be the number of elements of S_i. Assume that there exists a monic irreducible polynomial $\pi \in k[t]$ of odd degree such that one of the class $[Q_i]$ is unramified at π and the corresponding specialisation is zero. Then $\mathbf{PGO}^+(A, \sigma)(F)/R$ is finite, and its cardinality is at most $2^{n_1+n_2}$.*

Using this corollary, it is easy to construct an infinite family of non-quasi-split adjoint groups G of type ${}^2D_3$ defined over $k(t)$ (but not over k) such that $G(k(t))/R$ is finite for an (almost) arbitrary field k.

Example 9. Let k be a field of characteristic different from 2 and let $F = k(t)$. Let $a, \alpha \in k^\times$ and let $\pi \in k[t]$ be a monic irreducible polynomial satisfying the following conditions:

(1) $(-1) \cup (a) \cup (\alpha) = 0$.
(2) The quaternion k-algebra (a, α) is not split over $\kappa(\pi)$ (in particular, (a, α) is not split over k, and therefore is not split over F, and $\alpha \notin k^{\times 2}$).
(3) There exists $b \in k$ such that $\pi(b)$ is a non-zero norm in $k(\sqrt{\alpha})$.

Let $Q_1 = (a, \alpha) \otimes_k F$, $Q_2 = (\pi, \alpha)$, $Q = (a\pi, \alpha)$ and $L = F(\sqrt{a\pi})$. Let $1, i, j, ij$ be the standard basis of Q and γ its canonical involution. Notice that Q is a division algebra, since $\partial_\pi([Q]) = \mathrm{Res}_{\kappa(\pi)/k}(\alpha) \neq 0$ (otherwise (a, α) would be split over $\kappa(\pi)$).

Let σ be the involution on $A = M_3(Q)$ adjoint to the skew-hermitian form $\langle j, -aj, i\rangle$ over (Q, γ). The skew-hermitian form $h' := \langle j, -aj\rangle$ has trivial discriminant and the corresponding adjoint involution σ' on $A' := M_2(Q)$ can be written

$$\sigma' \simeq \sigma_{\langle 1, -a\rangle} \otimes \rho,$$

where ρ is the orthogonal involution on Q defined by

$$\rho(1) = 1, \qquad \rho(i) = i \quad \text{and} \quad \rho(j) = -j.$$

It is then easy to check that $C(A', \sigma') = Q_1 \times Q_2$, using the formulas describing Clifford algebras of tensor products of involutions (see [9, p. 150], for example, or [13]), and the fact that $\mathrm{disc}(\rho) = \alpha \in F^\times/F^{\times 2}$.

Claim. $\mathbf{PGO}^+(A,\sigma)(F)/R \simeq \mathbb{Z}/2\mathbb{Z}$.

Indeed, $[Q_1]$ has no pole and $[Q_2]$ has exactly one pole. Notice also that π is not a scalar multiple of $t-b$, since $\pi(b)\neq 0$ by assumption. Hence $[Q_2]$ is unramified at $t-b$. Moreover, we have $s_{t-b}([Q_2]) = (\pi(b)) \cup (\alpha) = 0$ by assumption. By Corollary 8, we then get that $|\mathbf{PGO}^+(A,\sigma)(F)/R| \leqslant 2$. Now it is enough to find a non-trivial-class in $\mathbf{PGO}^+(A,\sigma)(F)/R$. First of all, we clearly have $-a\pi \in N_{L/F}(L^\times)$. Moreover, since $(-1)\cup(a)\cup(\alpha)=0$, we have $-1 \in \mathrm{Nrd}_{Q_1}(Q_1^\times)$, so $a = (-1)\cdot(-a) \in \mathrm{Nrd}_{Q_1}(Q_1^\times)$. Since $-\pi \in \mathrm{Nrd}_{Q_2}(Q_2^\times)$, we get $-a\pi = a\cdot(-\pi) \in N_{L/F}(L^\times) \cap \mathrm{Nrd}_{Q_1}(Q_1^\times)\cdot \mathrm{Nrd}_{Q_2}(Q_2^\times)$. It remains to show that the R-equivalence class of $-a\pi$ is not trivial. For, it suffices to prove that $\varphi([-a\pi]) \neq 0$; this is the case since $\partial_\pi((-a\pi)\cup[Q_1]) = (a,\alpha)_{\kappa(\pi)} \neq 0$.

Remark 10. The group $\mathbf{PGO}^+(A,\sigma)$ obtained is not quasi-split since Q is a division algebra. Moreover, it is not defined over k. Otherwise $[Q]$ would be unramified at π, which is not the case as we have seen above. To obtain concrete examples, one may take for k any field such that $-1 \in k^{\times 2}$ such there exists a non-split quaternion algebra (a,α) over k, and for π any arbitrary monic irreducible polynomial of odd degree satisfying $\pi(0)=1$.

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